Diversitas Journal
ISSN 2525-5215
Volume 9, Issue 4 (Oct/Dec. 2024) p. 1892 – 1912
https://diversitasjournal.com.br/diversitas_journal

Fundamentals of Mathematics: Logical-Philosophical Concerns

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A B S T R A C T

This literature review presents a systematization of the concepts that underlie mathematics, showing the connections between the philosophy of science and logic. To conduct this literature review, databases such as Scopus, Web of Science and Google Scholar were consulted, aiming to obtain a broad and robust coverage of the fundamental concepts that relate mathematics to the philosophy of science and logic. It was argued that the teaching of Mathematics can advance in contributions to the philosophical basis so that the student realizes that this science is dynamic, and that its nature is composed of elements that allow for logical-philosophical debate, providing yet another alternative for the construction of knowledge.

R E S U M O

O presente trabalho de revisão de literatura apresenta uma sistematização dos conceitos que fundamentam a matemática, mostrando as conexões entre a filosofia das ciências e a lógica. Para realizar esta revisão de literatura, foram consultadas bases de dados como Scopus, Web of Science e Google Scholar, visando obter uma cobertura ampla e robusta dos conceitos fundamentais que relacionam a matemática com a filosofia das ciências e a lógica. Argumentou-se que o ensino de Matemática pode avançar em contribuições na base filosófica para que o aluno perceba que essa ciência é dinâmica, e que sua natureza é composta de elementos que permitem o debate lógico-filosófico proporcionando mais uma alternativa para a construção do conhecimento.

INFORMAÇÕES DO ARTIGO

Histórico do Artigo:
Submetido: 25/10/2023
Aprovado: 04/11/2024
Publicação: 07/11/2024

Keywords*:*
Fundamentals of Mathematics, Set Theory, Logic, Philosophy

Palavras-Chave:
Fundamentos da Matemática, Teoria de Conjuntos, Lógica, Filosofia

DOI: 10.48017/dj.v9i4.2837

Introduction

What is Mathematics? What are the contributions of Zermelo, Fraenkel, Neumann, and Gödel to the formation of mathematical thought? We know that Mathematics is the result of the union of several components that are rarely presented in a connected way and in the same literary work. These components need to be extracted and modeled from the philosophy of science, the philosophy of mathematics, logic, and, of course, own mathematics.

A large part of the history of Mathematics is made by anonymous people, contributions from philosophers and mathematicians from the past and present who face challenges in the obstinate search for demonstrations.

Mathematics is like bricks in a large construction, but its foundation is based on some components structured throughout history (BALESTRI, 2020). The interdisciplinarity between mathematics and philosophy is the key to the process, but addressing connections between these worlds requires a vast historical detail of the different schools of thought such as logicism, intuitionism, formalism and presenting all elements necessary for a foundation of Mathematics.

For Deleuze and Guattari (2010), all forms of knowledge are forms of thought, philosophy produces thought, but this is not the privilege of philosophy, Mathematics is knowledge that converges to the idea of a component placed by these philosophers, but not assimilated by them, because there is a distinction between forms of creation that characterize various types of knowledge. Following these steps, we present a systematization of concepts that describe and/or define the most common and widespread foundation of Mathematics in mathematics and logic books, organized philosophically.

For this article, studies were selected that address the dynamic character of mathematics and its philosophical connections, providing an overview of how logic and philosophy contribute to the understanding of mathematical concepts. We argue that the teaching of Mathematics can benefit from a philosophical basis, helping students to perceive mathematics as a dynamic science, with elements that enable a logical-philosophical debate. This approach, therefore, offers a pedagogical alternative that contributes to the construction of knowledge in a broader and more interactive way.

Mathematics, in its essence, is a science marked by abstraction, formalization and logical rigor, and its history is deeply intertwined with the development of philosophical thought. Since ancient times, questions about the nature of numbers, geometric figures and the very structure of mathematical reasoning have aroused the curiosity of philosophers and mathematicians, who sought to understand not only "How" to do mathematics, but "What" mathematics is and “What is its role in human knowledge?” Greek philosophers, such as Plato and Aristotle, took first steps in this direction, questioning whether mathematical objects existed independently of the physical world or whether they were creations of the human mind.

During the Middle Age, mathematics remained largely associated with logic and natural philosophy, gaining status and relevance in European universities. In the Modern Age, with the emergence of figures such as Descartes, Newton and Leibniz, mathematics became an essential pillar for science, being fundamental for the development of the scientific method. In this period, philosophical questions about the foundation of mathematics began to deepen, involving the need to clearly define mathematical concepts and the relationship between mathematics and observable reality. The search for a consistent and complete mathematical system, capable of explaining universal truths, culminated in a series of debates and studies that shaped modern science.

In the 19th century, mathematics underwent intense formalization, mainly in areas of logic and arithmetic, marking the beginning of the so-called "fundamental crisis". Theorists such as Frege, Russell and Hilbert dedicated themselves to systematizing mathematics, seeking to establish a solid and universal logical basis. This search culminated in the emergence of philosophical schools such as logicism, formalism and intuitionism, each offering different answers to the fundamental questions of mathematics. However, the work of Kurt Gödel, with his famous incompleteness theorems, shattered hopes for a fully closed and self-consistent mathematical system, revealing the limits of logical foundations and introducing new questions about the nature of truth and proof in mathematics.

In the contemporary context, mathematics continues to be a field of intense philosophical discussions. The nature of mathematical concepts, the validity of non-strictly logical methods and the applicability of mathematics to other sciences are topics that remain under debate. At the same time, these discussions have important repercussions for the teaching of mathematics, suggesting that, more than a practice of formulas and algorithms, mathematics can be understood as a dynamic activity, which involves critical thinking, debate and constant reflection on its structure itself.

Given this scenario, this work aims to present systematic definitions of foundations of Mathematics, showing connections between the philosophy of science and logic. From a literature review, it was demonstrated how these philosophical bases can enrich the teaching of mathematics, helping to demystify its image as a rigid and static science and encouraging a deeper and more dynamic understanding of its nature.

Methodological Path

To carry out this literature review, we adopted a systematic bibliographic review methodology. This approach involved the careful selection of studies and publications that address mathematics in its philosophical and logical bases, aiming to systematize fundamental concepts of this science and its interrelations with the philosophy of science and logic.

The Scopus, Web of Science and Google Scholar databases were searched, chosen for their scope and relevance in the area. On these bases, we used keywords such as "philosophy of mathematics", "mathematical logic", "epistemology" and "mathematics teaching" to identify publications aligned with research objectives.

Study inclusion criteria focused on publications that explored conceptual elements of mathematics from a philosophical perspective, in addition to research that discusses the role of logic and epistemology in the construction of mathematical knowledge. We excluded studies that did not present consistent theoretical discussions or that were restricted to empirical analyzes without philosophical foundations.

After selection, texts were analyzed to extract and organize keyconcepts that support mathematics as a dynamic science capable of logical-philosophical debates. We argue that this perspective can enrich the teaching of Mathematics, by allowing students to understand its constantly developing nature and its multiple philosophical connections, creating an environment more conducive to the construction of knowledge.

Component 1: Language

The journey begins with the definition of a language capable of "speaking" about mathematical objects with rigor and without ambiguities that a natural language has. For Deleuze and Guattari (1995, p.7-8),

> The elementary unit of language — the statement — is the command word. More than common sense, a faculty that would centralize information, it is necessary to define an abominable faculty that consists of issuing, receiving and transmitting command words.

In this sense, with language being the first component of foundations in Mathematics, it implies that from there the direction will be given for the concreteness of objects that are purely abstract. Symbols that define mathematical language are just ideas that materialize in the world.

Still according to Deleuze and Guattari (2010, p. 23),

> There is not simple concept. Every concept has components, and is defined by them. So there is a number. It is a multiplicity, although not all multiplicity is conceptual. There is no single-component concept: even the first concept, the one with which a philosophy "begins", has several components, since it is not evident that philosophy must have a beginning and that, if it determines one, it must add to it a point of view or a reason.

The discipline that deals with signs that constitute a language is called semiotics and semiosis is the use of these signs. Semiotics has three dimensions: "1. The signs themselves; 2. The objects designated by the signs; 3. The people who employ the signs." (COSTA, 1992, p. 69). Furthermore, with reference to signs and semiosis, signs, at least in complex cases, involve three types of relationships: they relate to objects, people and other signs (COSTA, 1992, p. 70).

Seeking to simplify concepts of semiotics, we can divide it into syntactics, semantics and pragmatics. Syntactics deals with the combination of language objects, that is, their morphology. Semantics concerns the meaning of these objects (interpretation of a word, a logical formula, etc.). And finally, pragmatics considers, in addition to the construction and meaning of symbols themselves, the people connected to semiosis.

In the context of Semiotics we need to "extract" elements necessary to designate mathematical objects. First steps in building a formal language were taken by Plato, Aristotle, Leibniz and Kant (SILVA, 2007). Gottlob Frege is a central figure due to his logicist project: Fundamentals of Arithmetic, launched in 1884, where he attempted to formalize mathematics.

Here we start talking about the language of set theory, which is first-order logic. It consists of an alphabet and formulas. The alphabet is made up of variables: lowercase letters x,y,z, ... which can be indexed by numbers; connectives: ⌐ (negation), → (conditional); quantifier: ∀ (universal quantifier); parentheses: these are the left and right parentheses and serve as punctuation; binary predicates: = (equality) and ∈ (belongs). Formulas are finite sequences of alphabet symbols and follow the following set of rules:

1. If x and y are variables, $x \in y$ and $x = y$ are formulas;
2. If L and M are formulas, $\neg(L), (L) \rightarrow (M)$ are formulas;
3. If L is formula and x is a variable, so $\forall x(L)$ is formula;
4. All formulas can be constructed using items 1, 2 e 3.

When we create formulas and they become extensive and/or complex, we need to add other symbols that can simplify these larger expressions. For example, we can simplify the formula ∀z((z∈x)→(z∈y)) as follows: x⊂y, which is saying that x is contained in y if every element of x belongs to y. Another way to simplify is by using constants, which correspond to proper names in natural language.

Completing simplification mechanisms, since we are using first order logic, others basic connectives can be deduced as follows:

- $\vee$ (disjunction): (A) $\vee (B)$ is an abbreviation of $(\neg(A)) \rightarrow (B)$;
- $\wedge$ (conjunction): $(A) \wedge (B)$ is an abbreviation of $\neg((\neg(A)) \vee (\neg(B))$;
- $\leftrightarrow$ (biconditional): $(A) \leftrightarrow (B)$ is an abbreviation fof $((A) \rightarrow (B)) \wedge ((B) \rightarrow (A))$;
- $\exists$ (existential quantifier): $\exists x(A)$ is an abbreviation of $\neg(\forall x(\neg(A)))$.

In 1929, Kurt Gödel demonstrated that first-order logic (the language of mathematics) is complete and consistent, that is, that all logically valid formulas can be derived within the system without the need to add new inference rules. The proof is known as Gödel's completeness theorem (ROQUE, 2012).

We saw here that we need to be very clear about the ideas we use in Mathematics, that often the simplification of terms is an attempt to make this language more accessible without losing its essence.

This language needs to be well defined so that there is no error in the idea of what we are talking about, for this reason standards and symbols were created for a so-called universal language.

Component 2: Definitions

For Scheinerman, "mathematical objects acquire existence through definitions. For example, a number is called prime or even as long as it satisfies precise conditions, without ambiguity" (SCHEINERMAN, 2011, p. 5). At this point, our search is for a component that has a legislative character, but which in the context of mathematics is obliged not to allow multiple interpretations of the same fact. If something is illegal in a given instance that concerns a subject or area of mathematics, then it will be illegal for all instances in any area of mathematics. Furthermore, the "critical" terms that constitute a given definition must first be defined. Other example: "An integer is called even if it is divisible by 2".

However, we understand, as in philosophy, that "there are not simple concepts. Every concept has components and is defined by them. So there is a number. It is a multiplicity, although not all multiplicity is conceptual" (DELEUZE; GUATTARI, 1992, p.23). In this definition of an even number, in addition to words from natural language, we have three terms that draw attention: integer, divisible and the number 2. When we talk about a foundation in Mathematics, what we mean is that it is necessary to have a base, a foundation from which everything else is built. In this sense, the number 2 is at the basis of the theory, since in the first stage it can be represented as a set, and sets, in turn, are primitive objects in ZFC (everything is a set) and are not defined. Their existence is guaranteed to the extent that they submit to the axioms of ZFC. Other than that, we are outside the rules of the "game" called Mathematics. So when it comes to definitions, according to Scheinerman (2011, p. 5),

> this is a game we cannot entirely win. If every term is defined in simpler terms, we will continually be searching for definitions. There must come a time when we say, "This term is indefinable, but we believe we understand what it means".

Searching to characterize the concept of definition within the scope of formal sciences, according to Sant'Anna (2005, p. XV),

> two classes of definitions are distinguished: abbreviative and amplificative. The first simply constitute processes that help to expose theories, not expanding their languages. They come in two categories: simple ones, which replace complex groups of symbols with a new symbol, and contextual ones, which introduce new symbols, such as abbreviations, in certain contexts. In principle, these definitions can be eliminated, as they are nothing more than auxiliary techniques in the construction of theories. As Russell said, these are typographic conventions.

To conclude studies on definition, we will quickly address fundamental concepts of three theories on the concept of definition contained in Wolenski and Kohler (1999).

1. Lesniewski's theory

To Lesniewski, every definition is only created when a new symbol is introduced into the language of a given theory. Thus, if we have a formula F that introduces a new symbol S into this theory, then two criteria must be satisfied:

- "Eliminability criterion: means that when writing a formula using a given defined concept, it can be rewritten in an equivalent way without any explicit mention of that concept" (SANT'ANNA, 2005, p. 18);
- "Non-creativity criterion: means it is impossible to obtain new results from the definition" (SANT'ANNA, 2005, p. 18).

2. Tarski's theory

In Mathematics, the concept of structure is defined as an ordered pair (set), where we have for the first element, a set and for the second, a set of relations. A collection of structures is called a species and it satisfies certain conditions that are the axioms of the species. For example, if the language to be worked on is first order (ZFC language). In Sant'Anna words (2005, p. 30), Let us call this language Λ. Let $e = \langle D, R \rangle$ be an interpretation for Λ. A set X of $\boldsymbol{e}$ is said to be definable according to Tarski if, and only if, there is a well-formed formula $\varphi(y)$ in Λ with only one free occurrence of a variable y, such that $x \in X$ if, and only if, x satisfies this formula. Therefore, we say that the formula φ(y) defines the set X.

3. Padoa Principle

Starting from the definitional concepts of Lesniewsli and Tarski, the Padoa principle according to Beth (1953, p. 330),

> Let S be an axiomatic theory whose primitive concepts (excluding logical constants) are $c_1, c_2, \dots, c_n$. Such concepts, as already mentioned, can be individual constants, relations, operations or sets. A given concept Such concepts, as already mentioned, can be individual constants, relations, operations or sets. A given concept c_i is independent (non-definable) of the concepts $c_1, c_2, \dots, c_{i-1}, c_i, c_{i+1,} \, c_n$ if, and only if, there are two models of S in which $c_1, c_2, \dots, c_{i-1}, c_i, c_{i+1,} \, c_n$ have the same interpretation; but the interpretations of c_i , in these models, are different.

These three theories show us that a definition "there are, most of the time, pieces or components coming from other concepts, which responded to other problems and assumed other plans" (DELEUZE; GUATTARI, 1992, p.26); and, in these convergences, we have connections where we group the definitions into what we call sets.

Component 3: Sets

Let us start discussing the notion of set in an intuitive way, in the sense that it is a primitive concept within the theory. So the idea, in principle, is to treat sets as any collection of objects or elements that are distinct. This conception is known as naive set theory (ROQUE, 2012) and was developed between 1874 and 1895 by the great mathematician Georg Ferdinand Ludwig Philipp Cantor.

According to Sant'Anna (2007, p. 1),

> The intuitive idea that a set is a collection of objects was nothing new. The surprise introduced by Cantor was the idea that infinities can be treated as well-defined, well-delineated objects, in some sense. In other words, Cantor was

> particularly interested in infinite sets, which even led him to classify different types of infinities.

For Cantor, the set is a non-enumerable compact grouping in which all points are accumulation and have an empty interior (BOYER; MERZBACH, 2012). Non-enumerable sets are those that do not have a bijection, that is, we cannot relate the elements of one set to another, for example, I have children in the park and not enough balls for them. But, Cantor worked with infinite sets.

According to Cantor, there are infinities greater than other infinities, everything would be explained by bijective correspondence, a part is not smaller than a whole, but if a set is finite (defined cardinality) the part is always smaller than a whole (DELAHAYE, 2006).

Cantor's definitions of infinity and the grouping into sets are similar to what Deleuze and Guattari (1995, p. 202) defended of infinity as the absolute horizon of the activity of thinking, as the unlimited and incommensurable, and also as the infinitely small within the that has limits, or as the infinitely variable from a finite set.

> We just ask for a little order to protect us from the chaos. There is nothing more painful, more distressing than a thought that escapes itself, than ideas that escape, that disappear poorly delineated, already gnawed into oblivion or precipitated into other ideas that we have not yet mastered. They are infinite variabilities whose disappearance and appearance coincide. They are infinite speeds that are confused with the immobility of the colorless and silent nothingness that they travel, without nature or thought. It is the moment in which we do not know whether it is too long or too short for time. We get eyelashes that crack like arteries. We incessantly lose our ideas. This is why we try so hard to maintain established opinions. We only ask that our ideas be concatenated according to a minimum of constant rules, and the association of ideas has never had any other meaning, to provide us with those rules of protection, similarity, contiguity, causality, which allow us to put a little order in our ideas. ideas, move from one to another according to an order of space and time.

At the beginning of the 20th century, serious problems relating to foundations of Mathematics emerged via paradoxes in both logic and set theory. Bertrand Russell (2007) began studying, in 1902, Gottlob Frege's work, Grundgesetze der Arithmetik, and this moment in the history of mathematics is crucial and had its main developments until the 1930s, but it is certainly still a key point in any study that seeks to formalize and/or present a Mathematics Foundation (ROQUE, 2012). Russel found a contradiction in the system proposed in Frege's work, which became known as “Russel's Paradox”. Let's get to it:

Let *z* be the set of all sets that are not members of itself. In Cantor's theory, we have: R={x|x∉x}. This set will lead to the following contradiction: R ∈R ↔ R∉R. Here, dear reader, we need a little logic to prove that we actually have a contradiction. Consider the following set:

R={x|x is regular}. Where regular = (x∉x). The following question then arises: Is R regular? We have two possible answers:

1. If $R \in R$, then $R \notin R$, as it does not have the property of being regular, which is not belonging to itself (x∉x)
2. If $R \notin R$, then $R \in R$, as it has the property of being regular, which is that it does not belong to itself (x∉x)

In both cases we have a contradiction. In the demonstration above, a small detail was purposely omitted. Let's make a new demo that fixes this flaw.

$R = \{x \mid x \text{ is a regular set}\}$. Let us ask the same question as in the previous demonstration.

1.1 If $R \in R$, then R is a set and is regular. By the connective "and" both statements must be true. This implies that item 1.1 leads to a contradiction, as we have R∉R, the same as occurred in item 1.

1.2. If $R \notin R$, then R is not set or is not regular. By the connective "or" one of the two statements must be false. By item 2, the second part of the statement (it is not regular) leads to R∈R and we also have a contradiction.

Having defined the nature of the set by Cantor, it is important to talk about David Hilbert (HILBERT, 1899), he is the greatest exponent in the defense of the formalization of mathematics. His work, which dealt with the foundations of geometry Grundlagen der Geometrie, published in 1899, makes clear the rigor he sought for mathematics. In short, his goal was to obtain an axiomatic system that should satisfy three conditions: consistency, completeness and independence.

Component 4: Axioms

Axiom is a dogma in Mathematics, a truth of faith. No demonstration needed. It is an extension of the previous item and comes to overcome paradoxes, especially Russel's, so we need to establish the conditions about the nature of a set and the ways in which we can construct them without paradoxes arising.

Cantor's pioneering ideas allowed results that went against intuition and brought critical problems to the foundations of mathematics (DAUBEN, 1990). His set theory was defined by the following axioms:

Axiom 1. Extensionality: A set is determined solely by its elements. So sets like $A = \{3,5,7,9\}$ and $B = \{5,7,3,9\}$ are equal. $A = \{1,2,3,4\}$ and $B = \{1,1,1,2,2,2,3,4\}$ so are equal. The order and repetition of elements are irrelevant.

Axiom 2. Comprehension: It guarantees that every logical formula φ(x) creates a new set. We denote the creation of a set C, as C={x|φ(x)}"."

Cantor's thought focused on what is real and what is illusion in Mathematics, in his set theory he rejected several axioms, because for him, as we see in Deleuze (1962, p. 118) "there is no truth that, before being true, is not the effectuation of a meaning or the realization of a value". His theory interprets propositions about mathematical objects such as numbers and functions, and provides a standard set of axioms to prove or disprove the data.

Zermelo-Fraenkel (ZF) set theory is a formal axiomatic theory. It is presented in a first-order language (see component 1) and all objects in it are sets. Zermelo and Fraenkel seek to map ideas presented in Cantor's theory, eliminating inconsistencies.

At this point, we need to clarify a question that is capable of showing more accurately the idea of a foundation of Mathematics. According to Sant'Anna (2007, p. 47),

> There are many formal set theories in the literature. Some are formulated in first-order languages, others in higher-order languages [...]. Still others are formulated in languages that do not use logical connectives, quantifiers or variables [...], while there are also those that are based on non-classical logic [...]. Ultimately, imagination is the limit. To illustrate the idea of the advantages of formalization, the most common formal set theory is discussed here, namely Zermelo-Fraenkel (ZF). In no precise sense is ZF better or worse than other known proposals. It is simply the most common because it is historically one of the first, allows a precise foundation of vast fields of mathematics, is very

intuitive compared to Cantor's original ideas and also because it is widely published in many important logic books.

The axiomatic set theory ZF together with the axiom of choice (Choice) is known as ZFC and in addition to having Zermelo and Fraenkel as the great creators, it had a decisive participation in its evolution and maturation by Thoralf Skolem, Dmitry Mirimanoff and Jonh Von Neumann . Let's go to the ZFC axioms:

1. Axiom of extensionality (ZFC1): every set is uniquely determined by its elements.

This axiom establishes a relationship between equality and relevance and explicitly talks about the nature of sets, comparison with other sets and brings with it other concepts. If two objects are not equal, then the following remains: they are disjoint, one is a sub-set of the other or there is an intersection between them. These three elements are related to the sense of relevance.

2. Foundation Axiom (ZFC2): "for every non-empty set A there is an element x disjoint from A" (SANTOS, 2007, p. 92).

The foundation axiom also deals with the nature of sets and prohibits the existence of certain strange "sets", an example of Cantor's theory and any set that meets this axiom is said to be well founded. Like this:

- $E = \{E\}$, $B = \{\{b\}, B\}$, are not sets in ZFC axiomatic theory;
- Infinite chains like $x \in x \in x \in x \cdots$ are also prohibite by ZFC2.

3. Substitution Axiom (ZFC3): "given a binary proposition π such that, $\forall x(\pi(x,y)=\pi(x,z)\rightarrow y=z)$ that is, the proposition defines a function whose domain is the set A, then the image of the set A by the function defined by π is also a set called the image of A by the function π." (SANTOS, 2007, p. 92). Thus we have a function π(x,y) that receives the elements of a set A (domain) and its image by the function defined by π is also a set. This axiom is actually a scheme of axioms, because for each function π we gain a new axiom.

The substitution axiom, given its complexity and application only in advanced topics of the theory, can be presented in a simpler version called the Separation Axiom (in ZFC it is a theorem). "Such a very cunning scheme of postulates was the solution that Zermelo found to avoid certain paradoxes, such as Russel's" (SANT'ANNA, 2007, p. 51). Zermelo's new version of Cantor's axiom 2 would look like this: $R=\{x \in A / \varphi(x)\}$. Here it is established that we can only build new sets from a pre-established set. Let's go back to Russell's paradox and see what happens: $R = \{x \in A \mid x \notin x\}$

▪ $R \in R$, so $R \in A$ and $R \notin R$. The second condition leads to a contradiction and as we have a conjunction, the entire sentence is invalidated;
▪ $R \notin R$, so $R \notin A$ or $R \in R$. The second condition leads to a contradiction, but we have a disjunction. Thus, the set R defined by the formula $x \notin x$ are all objects that are not in A. Using the axiom of separation it is possible to prove that the universe set is not a set in ZFC and also show the existence and uniqueness of the empty set ($\emptyset$), which in many works is presented as an axiom (Axiom of Emptiness). Here it is a theorem in ZFC.

4. Power Axiom (ZFC4): guarantees the existence of all the parts of a given set. This new set is denoted by $R = \mathcal{P}(A)$. For example, if $A = \{1,2,3\}$, so $\mathcal{P}(A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$. This axiom begins to clarify the "nature" of Mathematics, showing that there are several levels of infinity. Another example: the set of natural numbers is enumerable and the set of real numbers is not enumerable, that is, it is not possible for there to be a bijection between these two sets.

With ZFC3 and ZFC4 we can deduce that given any two sets we can form a new set that has these two sets as elements. For example, if a and b are sets, then {a, b} is also a set. In most books that deal with the topic, this construction of sets is presented as the Pair Axiom. Here it was introduced as a theorem in ZFC.

5. Union Axiom (ZFC5): "For every set, its union (the collection of all its members) is a set." (TSOUANAS, 2021, p. 521). To better understand this axiom, we will show two examples:

▪ If $R = \{1, \{2,3\}, 4, \{5,6,7\}\}$ so $A = \cup (R) = \{2,3,5,6,7\}$
▪ If $R = \{1,3,5,7,9,11,13\}$ so $A = \emptyset$. R does not have members of members as elements. With the ZFC axioms presented so far, we can define the intersection and union between sets. Furthermore, using the axioms of Union, Power and Separation it is possible to define the Cartesian product between any two sets of ZFC.

6. Infinite Axiom (ZFC6): "There is a set that has $\emptyset$ as a member and is closed for the operation x^+" (TSOUANAS, 2021, p. 535). Here we need to clarify this "thing" denoted by x^+. The successor set of a set x is defined as: $x^+ \overset{\text{def}}{=} x \cup \{x\}$. This definition is due to Von Neumann.

▪ The axiom of infinity allows the construction of sets of the form:

1. $\emptyset$
2. $\emptyset \cup \{\emptyset\} = \{\emptyset\}$
3. $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$

- These sets are the bricks for building the Naturals, that is, the numbers

as we know are a representation of sets within ZFC. Thus, we can simplify the above sets as follows:

1. $0 \overset{\text{def}}{=} \emptyset$
2. $1 \overset{\text{def}}{=} 0^+ = \emptyset^+ = \emptyset \cup \{\emptyset\} = \{\emptyset\}$
3. $2 \overset{\text{def}}{=} 0^{++} = \emptyset^{++} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$

This implies that in ZF there is a set whose elements are 1, 2, 3, 4, 5, 6, ... This set is infinite according to Dedekind's definition: “Let there be a set A. we call Dedekind's A- infinite if and only if it can be 'injected' into a proper subset of it.”(TSOUANAS, 2021, p. 535).

Furthermore, according to Coniglio (1997, p. 46),

> It is essential to realize that this axiom clearly separates Arithmetic (which can be performed without assuming the existence of infinite sequences) from other disciplines in advanced Mathematics, such as Analysis, which make essential use of the Axiom of Infinity.

7. Choose Axiom (ZFC7): “given a set z whose elements are non-empty and two-by-two disjoint, then there exists a choice set u, which has exactly one member in common with each element of *z*” (CONIGLIO, 1997, p. 48)

Trying to explain, according to Sant’Anna (2007, p. 57),

> Colloquially speaking, if *x* is a set whose elements are non-empty and disjoint sets when taken two by two, then there is a set y formed as follows: from each element of *x* “take” one and only one element arbitrary w to become an element of y. That is why we say that the intersection between y and each element *z* of x is a unitary set $\{w\}$.

- Note that, although *u* “chooses” an element from each member of z, the axiom has nothing to do with the existence of an effective procedure for making this choice” (CONIGLIO, 1997, p. 48).
- ZFC3 presents a function (property) φ that extracts the members of a set that satisfies this property and constructs a new set. ZFC7 seems to complement this axiom, in the sense that the φ function needs to be described (expressed) and the choice function does not.
- The principle of good order is equivalent to ZFC7: Every set admits a good order.

Searching to present an overview regarding the nature of these axioms, we can classify them as follows:

▪ Axioms that say about the behavior and/or nature of a set: ZFC1 e ZFC2;
▪ Axioms that allow building new sets: ZFC3, ZFC4, ZFC5 and ZFC7 (although it has a highly non-constructive character);
▪ Axiom that guarantees the existence of certain sets: ZFC6

The tension between Cantor's axiomatic set theory and other versions of set theory, such as that of Zermelo and Frankel, can be found, each with its virtues and defects, however it is undeniable that Cantor was essential for the foundations of Mathematics. All the persecution by his former advisor Leopold Kronecker meant that he was never recognized during his lifetime and died of starvation during the First World War (IEZZI, 2004).

Component 5: Theorems

In natural language we speak many types of phrases (sentences). Sometimes we give an order, other times we ask a question, and sometimes we make statements about a particular subject. The part we are interested in here are the declarative statements. “A theorem is a declarative statement about mathematics for which there is a proof” (SCHEINERMAN, 2011, p. 10).

The way theorems are presented follows a logical structure and is the clearest materialization (insertion) of first-order logic within mathematics content, with the exception of its axioms which are presented in formal language. The vast majority of theorems have some of following forms:

▪ If A (hypothesis), so B (conclusion). We have here the implication;
▪ If A, so B and if B, so A. We have a double implication.

Within a theory built on a formal system, we have:

▪ Definition 1: “A theorem of a formal theory $\mathcal{L}$ is the last formula B of a proof. Such a demonstration is called a demonstration or proof of *B*” (SANT’ANNA, 2005, p. 89).
▪ Definition 2: “A formal theory $\mathcal{L}$ is said to be decidable if there is an effective procedure for deciding whether a given well-formed formula of $\mathcal{L}$ is a theorem of $\mathcal{L}$. If this effective procedure does not exist, then the theory is said to be undecidable”(SANT’ANNA, 2005, p. 89).

Component 6: Demonstration

Searching to create a hierarchy, we first conceptualize the definitions and then make statements (theorems, postulates, etc.) about the objects “inside” mathematics. We need within this world to establish what is true and what is not. Truth in mathematics has particularities that distinguish it from other sciences. For example, in a legal system of justice, truth is established through a trial and its definition comes via a jury and/or judge. Still in this sense, we can have the truth (consensus) established through experiments. We can have the philosophical truth that is something unalterable under any circumstances, which states what it is:

> Dionysius affirms everything that appears, ‘even the harshest suffering’ and appears in everything that is affirmed. The multiple or pluralistic statement, this is the essence of the tragic. We will understand better if we think about the difficulties we encounter in making everything an object of affirmation (DELEUZE, 1962, p. 19).

At the same time, philosophically, the statement alone is not enough, there is a need for a search for what is hidden behind appearances or so-called true statements, which leads to Deleuze's conclusion: “The complete formula of the statement is: the whole, yes , universal being, yes, but universal being is said to be a single becoming, the whole is said to be a single moment” (DELEUZE, 1962, p. 81-82). Therefore, philosophical truth does not have a single meaning, nor is it static and definitive, being influenced by several other factors.

In mathematics we have demonstration (proof). To situate the issue, let's analyze Goldbach's conjecture: Every integer greater than 2 is the sum of two prime numbers. A project from the University of Aveiro, Goldbach conjecture verification (OLIVEIRA E SILVA; HERZOG; PARDI, 2018), has already confirmed the conjecture up to numbers of the order of 4·10^17, in other words we have an extremely successful experiment and everything indicates that the conjecture is in fact true. Returning to the statement above, we have the word “todo” and it brings us to the idea of ∀ (universal quantifier – “for everything”). Here lies the difference between what is acceptable as proof in mathematics compared to other sciences.

Demonstration types:

1. Direct Demonstration: “consists of, starting from the propositions $\alpha_1, \ldots, \alpha_1$ in a model $\mathfrak{M}$, to use the rules of inference and the rules of equivalence until you arrive at the proposition β” (SANTOS, 2007, p. 74)
2. Contrapositive Indirect Demonstration: “consists of starting from the premise $\neg\beta$ in a

model $\mathfrak{M}$ and using rules of inference and the rules of equivalence arrive at the argument $\neg\alpha_1 \dots \vee \neg\alpha_n$" (SANTOS, 2007, p. 75)

3. Indirect Demonstration by Reduction to the Absurd: "in a model $\mathfrak{M}$ consists of demonstrating, using the rules of inference and the rules of equivalence, that $\alpha_1 \wedge \cdots \wedge \alpha_n \wedge \neg\beta$ is a contradiction" (SANTOS, 2007, p. 75).

We can ask ourselves if there is a limit to what we can prove or is there knowledge that we are not allowed to have? To answer a mathematical question, there must be someone who has all the mathematical knowledge about that object. Mathematicians believe that at a given moment you have enough mathematics to solve all questions. This is what Hilbert advocated.

Component 7: Incompleteness

Between Hilbert's famous phrase, given in a lecture at the University of Munster in 1926, where he stated that "No one can expel us from the Paradise that Cantor created" and Gödel's Incompleteness Theorems, only 4 years passed. Hilbert wanted to express the following with this phrase: that it was possible to axiomatize the entire structure of mathematical knowledge and prove, by strictly finite means, that this axiomatic is consistent (does not produce contradictions).

Gödel studied logical systems in an abstract way, but they applied to what Hilbert wanted to do, and it was demonstrated that in a consistent logical system, there will be statements that can neither be proven nor disproved. For him, there are statements that are neither true nor false.

Logician Jakob Hintikka called the moment when Gödel announced the Incompleteness Theorems "Gödel's Sternstunde" (his star hour), the lecture took place on October 7, 1930 at a congress in Königsberg.

According to Goldstein (2008, p. 70),

> Gödel gave no sign of the revolution he was hiding under his sleeve until the last day of the congress, reserved for the general discussion of the articles from the previous two days. He waited until the general discussion was well advanced, and then mentioned, in a single perfect sentence, that true but non-deducible arithmetical propositions were possible, and he had proved that they existed.

Let us look at Gödel's famous theorems, according to Carnielli, Rathjen (1990, p. 4),

> 1st Incompleteness Theorems: In every consistent formal system S, with a minimum of arithmetic, it is possible to formalize a sentence U such that U can be intuitively interpreted as the statement that it itself is unprovable in S.

> 2nd Incompleteness Theorems: The proof of consistency for formal systems (under the conditions that Hilbert wanted) cannot be formalized within the system itself.

We have a foundation in Mathematics, but there are statements that follow the rules of the game and where it cannot be proven that they are true or false. For example, Goldbach's conjecture, presented above, may be one such statement. Another famous problem that has so far escaped being proven or disproved is the Riemann hypothesis.

To conclude

After this review presenting a systematization with seven components that dealt with elements of philosophy, logic and mathematics, we can then make all necessary connections to glimpse the foundations of Mathematics and thus outline a reasonable answer that achieves the central purpose of this article.

Presented in a first-order language and based on axiomatic set theory, Mathematics communicates the objects of its theory through its definitions and evolves and/or consolidates its fields of action as new theorems are demonstrated, although it is not capable of proving its own consistency and being "incomplete", as it harbors in its "world" objects (statements) whose truth or falsity nothing can be said due to Gödel's incompleteness theorems.

To show how objects are constructed in mathematics using the concepts covered in this article, we will answer the following question: What is a function? We have been working with this definition since elementary school. Let's sketch a demonstration:

1. Usando os axiomas de ZFC, podemos construir o conjunto $\{x, y\}$;
2. Por ZFC1 sabemos que $\{x, y\} = \{y, x\}$, então precisamos incrementar algo a mais;
3. Definimos um par ordenado como $(x, y) \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$, que é conhecido como par de Kuratowki;
4. Usando dois conjuntos A e B, onde podemos formar pares ordenados $\{x, y\}$, com x sendo elemento de A e y elemento de B. O conjunto de todos os pares ordenados é chamado de produto cartesiano e definido como: $A \times B \stackrel{\text{def}}{=} \{(x, y) \ / \ x \in A \wedge y \in B\}$;
5. Uma relação (R) é um conjunto de pares ordenados e é chamada de relação binária entre dois conjuntos A e B, se R é um subconjunto de $A \times B$ (ex.: as desigualdades $<$, $>$ são relações);
6. Uma função é uma relação com uma propriedade bem especial, a saber:

Uma relação R entre dois conjuntos A e B, onde para todo $(\forall)$ $x \in A$ existe um único $y \in B$ e $(x, y) \in R$

Here, once again, the meaning of a Mathematics Foundation becomes clear: A mathematical object such as a function represents a certain set within ZFC. See that we used several of keywords in this article and reduced the notion of function to the level of a set, where all mathematics is based.

Hilbert said that "we hear within us the perpetual call: There lies the problem. Look for your solution. You can find it through pure reason, because in mathematics there is no *ignorabimus*" (HILBERT, 1899). I could end this story with this feeling of sadness or defeat, as Kurt Gödel proved that "ignorabimus" exist, but we can take the path that leads back to Gödel himself, as he had the nickname "Mister Why", and the essence of Learning comes through questioning. However, it is worth remembering that squaring the circle took more than two thousand years to be resolved.

Mathematics is a mystery, and these themes of inconsistency bring great philosophical concerns, for example, Galileo believed that nature was described in mathematical language. Does Mathematics itself impose obstacles to its own knowledge? Or have men not attained sufficient knowledge to demonstrate it completely?

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